Ricci limit flows and weak solutions

Robert Haslhofer

University of Toronto

joint with: Beomjun Choi

September 2024

Motivation and History

Consider a Ricci flow $(M, (g_t)_{t\in[0,T)})$ on a compact manifold M:

$$\partial_t g_t = -2\mathrm{Rc}(g_t)$$

By classical theory (Hamilton, DeTurck, Sesum), there exists a unique smooth solution on a maximal time interval $[0, T)$, characterized by

$$\limsup_{t\nearrow T} \max_{p\in M} |\mathrm{Rc}(g_t)(p)| = \infty$$

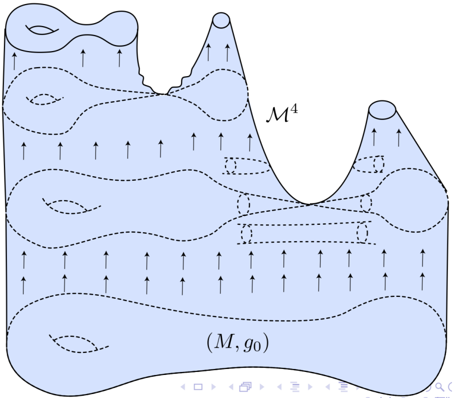
Questions:

- Continue flow beyond first singular time?
- Compactness theorem?
- Regularity and structure theory for weak limits?
- Notion of weak solutions?
- Existence and uniqueness?

The 3-dimensional case

Theorem (Perelman, Kleiner-Lott, Bamler-Kleiner)

For any compact 3-manifold (M, g_0), there exists a unique Ricci flow through singularities with initial condition (M, g_0).



The 3-dimensional case (continued)

Further properties:

- good compactness and partial regularity theory (in particular, the flow is given by a smooth compact manifold at almost every time, and is uniquely characterize by its behaviour on the smooth part)
- great topological and geometric applications (Bamler-Kleiner):
  - proof of generalized Smale conjecture: $\mathrm{Diff}(S^3/\Gamma) \sim \mathrm{Isom}(S^3/\Gamma)$
  - contractibility of the space of PSC metrics: $\mathrm{Met}_{\mathrm{PSC}}(M^3) \sim *$

Remark: The 3-dimensional theory can be generalized to higher dimensions assuming positive isotropic curvature (Chen-Zhu, Brendle, H)

Examples in higher dimensions

- Examples of 4-dimensional Ricci flows, whose blowup limits at the singularity are: Eguchi-Hanson, $\mathbb{R}^4/\mathbb{Z}_2$ (Appleton)
- Examples of non-uniqueness of Ricci flow through conical singularities in dimension $n \geq 5$ (Angenent-Knopf)
- Examples of Ricci flows in dimensions $n \geq 13$ that form type II singularities, where the scalar curvature is expected to be bounded, and that are modelled on Ricci-flat cones (Stolarski)
- Examples of Kähler-Ricci flows that develop a singularity that cannot be modelled on any smooth shrinking soliton (Li-Tian-Zhu)

Theory of Ricci flow in higher dimensions?

Recent developments:

- Compactness and partial regularity theory (Bamler)
- Proposed notion of weak solutions (H-Naber)

Goals for todays talk:

- Give introduction to Bamler's theory.
- Briefly recall approach from H-Naber.
- Reconcile the two approaches.

Recall: Gromov-Hausdorff limits of Einstein metrics

Consider a sequence of Einstein manifolds $\mathrm{Rc}(g_i) = \lambda_i g_i$, where $|\lambda_i| \leq 1$, on M_i^n. Then (looking around any choice of base-points $p_i \in M_i$) a subsequence *Gromov-Hausdorff converges* to a metric length space:

$$(M_i, g_i, p_i) \to (X, d, p)$$

Under the *noncollapsing condition* $|B(p_i, 1)| \geq v > 0$, the volume measures Vol_{g_i} converge to $\mathcal{H}^n_X$ (Colding), and there is a regular-singular decomposition

$$X = R \cup S,$$

such that:

- R is an open manifold with a smooth Einstein metric g, such that (X, d) is isometric to the metric completion of (R, g),
- $\dim S \leq n - 4$ (Cheeger-Naber),
- All tangent cones are metric cones (Cheeger-Colding).

Informal overview of Bamler's theory

Bamler developed a compactness and partial regularity theory for the Ricci flow that is comparable to (and implies) that of Einstein metrics.

In particular, he:

- introduced a parabolic version of metric spaces called "metric flows", and a parabolic version of Gromov-Hausdorff convergence.
- proved that every sequence of n-dimensional Ricci flows has a convergent subsequence that converges to a metric flow $\mathcal{X}$.
- under a noncollapsing condition (which is perfectly natural by Perelman's monotonicity formula) proved that there is a regular-singular decomposition $\mathcal{X} = \mathcal{R} \cup \mathcal{S}$, such that:
  - $\mathcal{R}$ is a smooth Ricci flow spacetime, and $\mathcal{R}$ uniquely determines $\mathcal{X}$,
  - $\dim \mathcal{S} \leq (n+2) - 4$,
  - All tangent flows are (possibly singular) shrinking solitons.

Heat kernel on Ricci flow background

$(M, g_t)_{t\in I}$ a Ricci flow: $\partial_t g_t = -2\mathrm{Rc}(g_t)$.

Heat kernel $K(p,t;q,s)$, where $p,q \in M$ and $s<t$ in I, is defined by

$$(\partial_t - \Delta_{g_t})K(\cdot,\cdot;q,s) = 0, \qquad \lim_{t\searrow s} K(\cdot,t;q,s) = \delta_q.$$

By duality, as a function of (q,s) it solves the adjoint problem

$$(-\partial_s - \Delta_{g_s} + R_{g_s})K(p,t;\cdot,\cdot) = 0, \qquad \lim_{s\nearrow t} K(p,t;\cdot,s) = \delta_p.$$

Adjoint heat kernel probability measures: $d\nu_{(p,t);s} = K(p,t;\cdot,s)d\mathrm{Vol}_{g_s}$.

Important properties of the heat flow

Reproduction formula: If $s < t' < t$ then

$$K(p,t;q,s) = \int_M K(p,t;\cdot,t')K(\cdot,t';q,s)d\mathrm{Vol}_{g_{t'}}$$

Sharp gradient estimate: $|\nabla u| \leq 1$ preserved under heat flow.

More precisely, if $u_{t_0} = \Phi_{t_0} \circ f_{t_0}$ for some 1-Lipschitz function f_{t_0}, then for all $t \geq t_0$ we have $u_t = \Phi_t \circ f_t$ for some 1-Lipschitz function f_t.

(Here, Φ_t is the solution of the 1d heat equation with $\Phi_0 = \chi_{[0,\infty)}$)

Metric flows

A **metric flow** $\mathcal{X} = (\mathcal{X}, \mathfrak{t}, (d_t)_{t\in I}, (\nu_{x;s})_{x\in\mathcal{X}, s\leq \mathfrak{t}(x)})$ consists of

- a set $\mathcal{X}$ (spacetime),
- a function $\mathfrak{t} : \mathcal{X} \to \mathbb{R}$ (time function),
- complete separable metrics d_t on the time-slices $\mathcal{X}_t = \mathfrak{t}^{-1}(t)$,
- and probability measures $\nu_{x;s} \in \mathcal{P}(\mathcal{X}_s)$,

such that the reproduction formula and the sharp gradient estimate hold.

Any smooth Ricci flow $(M, (g_t)_{t\in I})$ can be viewed as metric flow via:

- $\mathcal{X} = M \times I$,
- $\mathfrak{t} =$ projection on 2nd factor,
- $d_t = d_{g_t}$ induced metric on time-slices,
- $\nu_{(p,t);s} =$ adjoint heat kernel measure based at $x = (p, t)$.

Monotone quantities for adjoint heat kernels

Wasserstein distance: $d_{W_1(g)}(\mu_1, \mu_2) = \sup_{|\nabla f| \leq 1} \int f d\mu_1 - \int f d\mu_2$

$s \mapsto d_{W_1(g_s)}(\nu_{x;s}, \nu_{y;s})$ is monotone (McCann-Topping)

Variance: $\mathrm{Var}(\mu) = \int \int d^2(x, y) d\mu(x) d\mu(y)$

$s \mapsto \mathrm{Var}(\nu_{x;s}) + H_n s$, where $H_n = 4 + (n-1)\frac{\pi}{2}$, is monotone (Bamler)

Roughly speaking, to prove his compactness theorem, Bamler takes a Gromov-Wasserstein limit of the metric measure spaces at a countable dense subset of times, and then fills in other times using monotonicity.

Noncollapsing and partial regularity

$\mathcal{N}_{(p,t)}(\tau) = -\int_M \log K(p,t;\cdot,t-\tau)\,d\nu_{(p,t);t-\tau} - \frac{n}{2}(1+\log(4\pi\tau))$

$\tau \mapsto \tau\mathcal{N}_{(p,t)}(\tau)$ is concave, and gives noncollapsing (Perelman)

Theorem (Bamler)

For any sequence of pointed Ricci flows $(M_i^n, g_i(t)_{t\in(-T_i,0]}, (p_i,0))$, a subsequence converges to a metric flow $\mathcal{X}$ over $(-\lim_{i\to\infty} T_i, 0]$.

If the noncollapsing condition $\mathcal{N}_{(p_i,0)}(\tau_0) \geq -Y_0 > -\infty$ holds, then we have a regular-singular decomposition $\mathcal{X} = \mathcal{R} \cup \mathcal{S}$ such that:

- *$\mathcal{R}$ is a smooth Ricci flow spacetime, and $\mathcal{X}$ is determined by $\mathcal{R}$.*
- *The parabolic $*$-Minkowski dimension of $\mathcal{S}$ is $\leq (n+2)-4$.*
- *All tangent flows of $\mathcal{X}$ are (possibly singular) shrinking solitons.*

Characterizations of the Ricci flow

Theorem (H-Naber)

A smooth time-dependent family $(M, (g_t)_{t\in I})$ evolves by Ricci flow if and only if for almost every (p,t) the infinite dimensional gradient estimate

$$\left|\nabla_p \mathbb{E}_{(p,t)}[F]\right| \leq \mathbb{E}_{(p,t)}[|\nabla^{\|} F|]$$

holds for all test functions $F(B) = f(B_{\tau_1}, \ldots, B_{\tau_k})$ on path-space.

Here, $\mathbb{E}_{(p,t)}$ denotes the expectation with respect to Brownian motion starting at (p,t) given by

$$\mathbb{P}_{(p,t)}[B_{\tau_1} \in U_1, \ldots, B_{\tau_k} \in U_k] = \int_{U_1\times\ldots\times U_k} d\nu_{(p,t);t-\tau_1}(p_1)\ldots d\nu_{(p_{k-1},t-\tau_{k-1});t-\tau_k}(p_k),$$

and $\nabla^{\|} F(B) = \sum_{i=1}^k P_{\tau_i}\nabla^{(i)} f(B_{\tau_i}, \ldots, B_{\tau_k})$, where P is stochastic parallel transport defined via Hamilton's space-time connection.

Ricci limit flows and weak solutions

Theorem (Choi-H)

Any noncollapsed limit of smooth Ricci flows, as provided by Bamler's precompactness theorem, is a weak solution in the sense of H-Naber.

More precisely, given any noncollapsed Ricci limit flow $\mathcal{X}$, for any regular point $x = (p, t)$ we have the infinite dimensional gradient estimate

$$\left|\nabla_p \mathbb{E}_{(p,t)}[F]\right| \leq \mathbb{E}_{(p,t)}[|\nabla^{\|} F|]$$

for all test functions $F(B) = f(B_{\tau_1}, \ldots, B_{\tau_k})$ on path-space.

In fact, our argument applies to any noncollapsed metric flow that satisfies Bamler's partial regularity and solves the equation on the smooth part.

Corollary (Choi-H)

Every singular Ricci flow (for $n = 3$, or for $n > 3$ assuming PIC) in the sense of Kleiner-Lott is a weak solution in the sense of H-Naber.

Brownian motion and small sets

Theorem (Kakutani 1944)

A closed set $A \subset \mathbb{R}^n$ does not get hit by Brownian motion if and only if it has zero Newtonian capacity. Namely:

$$\mathbb{P}_x[B_t \in A \text{ for some } t > 0] = 0 \quad \Leftrightarrow \quad \mathrm{Cap}_N(A) = 0.$$

Newtonian capacity:

$$\mathrm{Cap}_N(A) = \inf \left\{ \int_{\mathbb{R}^n \setminus A} |\nabla u|^2 \quad \middle| \quad u = 1 \text{ on } A, u \to 0 \text{ at } \infty \right\}$$
$$= \left[\inf_{\mu(A)=1} \int_{A \times A} \frac{1}{|x-y|^{n-2}} d\mu(x) d\mu(y) \right]^{-1}$$

This is a great theorem, but one drawback is that it cannot be upgraded to a quantitative estimate.

The Benjamini-Pemantle-Peres estimate

Theorem (Benjamini-Pemantle-Peres 1995)

For any closed set $A \subset \mathbb{R}^n$*, where* $n \geq 3$*, we have*

$$\frac{1}{2}\mathrm{Cap}_M(A) \leq \mathbb{P}_0[B_t \in A \textit{ for some } t > 0] \leq \mathrm{Cap}_M(A),$$

where the Martin capacity is defined by

$$\mathrm{Cap}_M(A) = \left[\inf_{\mu(A)=1} \int_{A\times A} \frac{|y|^{n-2}}{|x-y|^{n-2}} d\mu(x) d\mu(y)\right]^{-1}$$

- the constants are sharp (consider spheres and spherical shells)
- the assumption $n \geq 3$ is natural, since planar Brownian motion will always hit with probability 0 or 1 (however, by killing the motion at a finite time, one can obtain a planar version of the theorem)

Proof of the BPP estimate – upper bound

Consider the stopping time $\tau := \min\{t > 0 : B_t \in A\}$. The distribution of B_τ on the event $\tau < \infty$ is a (possibly defective) distribution ν satisfying

$$\nu(A) = \mathbb{P}[\tau < \infty] = \mathbb{P}[\exists t : B_t \in A].$$

Now, recall the standard formula $\mathbb{P}\left[\exists t > 0 : |B_t - y| < \delta\right] = (\delta/|y|)^{n-2}$. By first entrance decomposition, this probability is at least

$$\mathbb{P}\left[|B_\tau - y| > \delta \,\text{and}\, \exists t > \tau : |B_t - y| < \delta\right] = \int_{|x-y|>\delta} \frac{\delta^{n-2}}{|x-y|^{n-2}}\, d\nu(x).$$

Dividing by δ^{n-2} and letting $\delta \to 0$, this yields

$$\int_A \frac{1}{|x-y|^{n-2}}\, d\nu(x) \leq \frac{1}{|y|^{n-2}}.$$

Hence, we conclude that

$$\mathrm{Cap}_M(A) \geq \left[\int_{A\times A} \frac{|y|^{n-2}}{|x-y|^{n-2}} \frac{d\nu(x)}{\nu(A)} \frac{d\nu(y)}{\nu(A)}\right]^{-1} \geq \nu(A).$$

Proof of the BPP estimate – lower bound

For $\delta > 0$ set $h_\delta(r) = (\delta/r)^{n-2}$ if $r > \delta$ and 1 if $r \leq \delta$.
Given any probability measure μ on A, consider the random variable

$$Z = \int_A 1_{\{\exists t>0: B_t \in B(x,\delta)\}} \frac{d\mu(x)}{h_\delta(|x|)}.$$

Clearly $\mathbb{E}[Z] = 1$. Now, we estimate the second moment:

$$\begin{aligned}
\mathbb{E}[Z^2] &= 2\mathbb{E}\int_A\int_A 1_{\{\exists t>0: B_t\in B(x,\delta) \text{ and } \exists s>t: B_s\in B(y,\delta)\}} \frac{d\mu(x)}{h_\delta(|x|)}\frac{d\mu(y)}{h_\delta(|y|)}\\
&\leq 2\mathbb{E}\int_A\int_A \frac{h_\delta(|y-x|-\delta)}{h_\delta(|y|)} d\mu(x)d\mu(y)\\
&\leq 2\mathbb{E}\int_A\int_A 1_{\{|y-x|\geq 2\delta\}}\left(\frac{|y|}{|y-x|-\delta}\right)^{n-2} d\mu(x)d\mu(y) + O(\delta)
\end{aligned}$$

Since $\mathbb{P}[\exists t > 0 : B_t \in A] \geq \mathbb{P}[Z > 0] \geq 1/\mathbb{E}[Z^2]$, we conclude that

$$\mathbb{P}[\exists t > 0 : B_t \in A] \geq \frac{1}{2}\mathrm{Cap}_M(A).$$

Hitting estimate for Ricci flow

The most important step for our proof is the following hitting estimate:

Theorem (Choi-H)

If $(M, g_t)_{t\in[t_0-2,t_0]}$ is a Ricci flow with Nash entropy bounded below, then

$$\mathbb{P}_{(p_0,t_0)}\Big[B_\tau \,\mathrm{hits}\, \mathcal{S}_\varepsilon \cap P^*(p_0,t_0,1)\,\mathrm{for\,some}\,\tau\in(0,1)\Big] \leq C\varepsilon^{2-\delta}.$$

- Here, the ε-singular set is defined by $\mathcal{S}_\varepsilon = \{(p,t) : \mathrm{reg}(p,t) \leq \varepsilon\}$, where $\mathrm{reg}(p,t) := \sup\{r \leq 1 : \sup_{P(p,t,r)} |\mathrm{Rm}| \leq r^{-2}\}$.
- Heuristically, one can of course easily guess the (almost) quadratic dependence on ε in light of Bamler's codimension 4 theorem.
- A fundamental new challenge for Ricci flow is that the heat kernel only has upper bounds, but no lower bounds. We compensate for the lack of lower heat kernel bounds, by making use of the space-time geometry, in particular heat kernel centers and P^*-parabolic balls.

Open problems

Existence

Can we construct a weak Ricci flow through singularities?

- Existence holds in dimension 3 by Perelman and Kleiner-Lott.
- In higher dimensions, given Bamler's compactness theorem, the key would be to come up with some suitable approximation scheme.

Uniqueness

Is weak Ricci flow through cylindrical singularities unique?

- Weak solutions provide a framework to discuss this question.
- Uniqueness holds in dimension 3 by Bamler-Kleiner.
- Examples by Angenent-Knopf show that Ricci flow through conical singularities can be nonunique.
- Question motivated by uniqueness of mean curvature flow through neck singularities proved by Choi-H-Hershkovits-White.